STEADY TWO-LAYER SOURCE-SINK FLOW

Lynne Talley

I. Introduction

Ocean circulation can be thought of as being forced almost entirely by heating and cooling, whether directly, as a result of heat transfer across the ocean surface, or indirectly by the winds which arise from heating and cooling of the atmosphere. In this paper we will be mainly interested in extremely idealized circulation produced directly by cooling and heating. Many simplifications are made with respect to the flow, the basin geometry and the type of forcing but it is hoped that insight will be gained into the circulation in regions where cooling and heating are particularly important. We specifically have in mind the circulation of the northern North Atlantic, the Norwegian-Greenland Sea and to a lesser extent, the Labrador Sea. The Norwegian--Greenland Sea is particularly well known as the source of the cold saline Bottom Water which enters the North Atlantic in deep western boundary currents and which contributes its characteristics to the North Atlantic Deep Water. Bottom Water is formed in the large cyclonic gyre occupying the Greenland Sea from inflowing Atlantic Water (Carmack and Aagaard, 1973) which enters the Norwegian Sea as the broad northward Norwegian Current. It subsequently appears to become more topographically controlled as it strengthens on the eastern flank of the Jan Mayen Ridge, flows through the gap in the East Jan Mayen Ridge and then along the eastern side of the Greenland Basin where it forms the eastern side of the cyclonic Greenland Sea gyre (Metcalf, 1960).

To some extent, there is a similar process in the Labrador Sea, although the dense water which is formed there is an Intermediate Water rather than

Bottom Water. There also, a cyclonic gyre is the scene of production of dense water, fueled by the inflow of Atlantic Water in the West Greenland Current and colder fresher water from the north.

The model discussed here is a steady extension of the time dependent two-layer model investigated analytically and numerically by Gill (1979b) and the time dependent axisymmetric model of Gill et. al. (1979). It is a steady two-layer model, intended for instance to model the upper Atlantic Water and deep Bottom Water of the Norwegian-Greenland Sea, in which cooling is introduced as simple mass and momentum transfer from a layer of density ρ_1, to a layer of density ρ_2. We will not concern ourselves with the actual mechanism for production of denser water, but rather with the resulting circulation. Steady linear solutions for the baroclinic mode will be sought for various types of distributed transfer in a meridional channel and then for point transfer in a horizontally infinite ocean, on the f and β planes, motivated by the apparent presence of large scale density currents and localized Bottom Water formation in the Norwegian-Greenland Sea. The effects of bottom friction and topography are not included.

Formulation in terms of a two-layer model is Largely motivated by the apparent two component nature of the Norwegian-Greenland Sea circulation. We undoubtedly lose some information about the vertical structure of the flow but can, nevertheless, see the broad outlines of the forced solution.

We will see that inclusion of diffusion in a steady two-layer model implies the possibility of eastern boundary layers in both f and β plane steady ocean circulation models.

II. Equations

The two-layer system is illustrated in Fig. 1. h_1 is the variable height of the upper layer, H_2 its rest value and $h = H_1 - h_1 = h_2$ the height of the interface above its resting value h = 0. ρ_1 is the density of the upper layer, p_1 its pressure and p the value of the pressure at the rigid lid. The depth integrated equations of motion and continuity for the two layers are:

$$\frac{Du_1}{Dt} - fv_1 = -\frac{p_x}{A} - \epsilon_1^*(u_1 - u_2) + \tau^{(x)}$$

$$\frac{Dv_1}{Dt} + fu_1 = -\frac{p_y}{\rho_0} - \epsilon_1^*(v_1 - v_2) + \tau^{(y)}$$

$$p_1 = p + \rho_1 g(H_1 + H_2 - z)$$

$$\frac{Dh_1}{Dt} + H_1(u_{1x} + v_{1y}) = -Q + \epsilon_2^* h \qquad (2.1)$$

$$\frac{Du_2}{Dt} - fv_2 = -\frac{p_x}{\rho_0} - g'h_x - \epsilon_1^*(u_2 - u_1)$$

$$\frac{Dv_2}{Dt} + fu_2 = -\frac{p_y}{\rho_0} - g'h_y - \epsilon_1^*(v_2 - v_1)$$

$$p_2 = p + \rho_2 g(H_2 + h - z) + \rho_1 g(H_1 - h)$$

$$\frac{Dh_2}{Dt} + H_2(u_{2x} + v_{2y}) = Q - \epsilon_2^* h$$

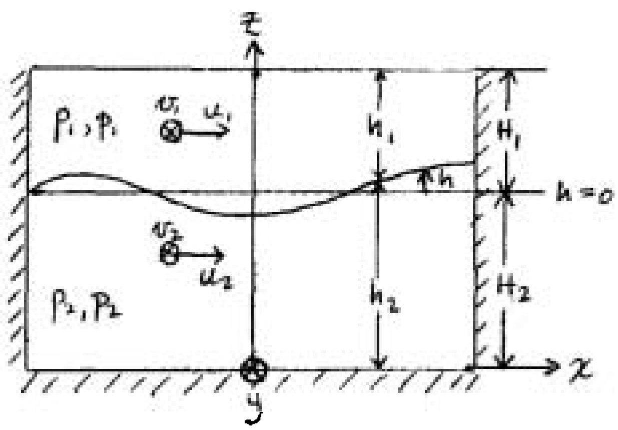

Fig. 1. Two-Layer Geometry.

We make the Boussinesq approximation and have already included hydrostatics in writing the x and y momentum equations where $g' = \frac{\rho_2 - \rho_1}{\rho_0} g$. Q is the mass transfer from the top to bottom layer, ϵ_1^* the coefficient of momentum transfer and ϵ_2^* the coefficient of diffusion. We have also included wind stress although, in the absence of bottom friction, the barotropic component of the flow can never be steady. The term $\epsilon_2^* h$ is absolutely crucial for 'the existence of steady solutions in the presence of a nonzero mass transfer Q since continual transfer without damping would imply continual spinup. One way of obtaining diffusion terms of this form is by using a normal mode analysis. If the buoyancy frequency N is constant, the variations in the vertical are sinusoidal with a fixed wave number m for each mode. Thus the operator

$$\frac{\partial}{\partial t} - K \frac{\partial^2}{\partial z^2}$$

can be replaced by

$$\frac{\partial}{\partial t} + Km^2$$

for this particular mode, and the latter operator is the one used here. The coefficient Km^2 varies from mode to mode, but here only one is considered. The same method can be used if N is not constant, but only if the diffusion coefficient varies with height in a suitable manner.

We look for steady and linear solutions. The appropriate scaling for the problem is

$$[x,y] \equiv R = \frac{\sqrt{g'H'}}{f} \qquad [p] = fUR\rho_0 \qquad [Q] = Q_0$$

$$[u_1, u_2, v_1, v_2] = U = \frac{Q_0 R}{H'} \qquad [h] = \frac{fUR}{g'} \qquad [\tau] = T_0$$

where $g' = g\frac{\rho_2 - \rho_1}{\rho_0}$ and $H' = \frac{H_1 H_2}{H_1 + H_2}$

The resulting nondimensional equations are

$$
\begin{aligned}
-v_1 &= -p_x - \epsilon_1(u_1 - u_2) + \frac{T_0}{fQ_0R}\tau^{(x)} \\
u_1 &= -p_y - \epsilon_1(v_1 - v_2) + \frac{T_0}{fQ_0R}\tau^{(y)} \\
u_{1x} + v_{1y} &= -\frac{H'}{H_1}Q + \frac{H'}{H_1}\epsilon_2 h
\end{aligned}
\qquad (2.3)
$$

$$
\begin{aligned}
-v_2 &= -(p+h)_x - \epsilon_1(u_2 - u_1) \\
u_2 &= -(p+h)_y - \epsilon_1(v_2 - v_1) \\
u_{2x} + v_{2y} &= \frac{H'}{H_2}Q - \frac{H'}{H_2}\epsilon_2 h
\end{aligned}
$$

where $\epsilon_1 = \frac{\epsilon_1^*}{f}$ and $\epsilon_2 = \frac{\epsilon_2^*}{f}$.

(If the primary driving force were the wind, velocity would be scaled according to wind stress T_o and not mass transfer Q_0).

We would like to look separately at the baroclinic and barotropic flows and therefore form the sum and difference equations with $\bar{\underset{\sim}{u}} = \underset{\sim}{u}_1 + \underset{\sim}{u}_2$ and $\hat{\underset{\sim}{u}} = \underset{\sim}{u}_2 - \underset{\sim}{u}_1$:

$$
\begin{aligned}
-\bar{v} &= -(2p + h)_x + \frac{T_0}{fQ_0R}\tau^{(x)} \\
\bar{u} &= -(2p + h)_y + \frac{T_0}{fQ_0R}\tau^{(y)} \\
H_1(u_{1x} + v_{1y}) &+ H_2(u_{2x} + v_{2y}) = 0
\end{aligned}
\qquad (2.4a)
$$

and

$$
\begin{aligned}
-\hat{v} &= -h_x - 2\epsilon_1\hat{u} - \frac{T_0}{fQ_0R}\tau^{(x)} \\
\hat{u} &= -h_y - 2\epsilon_1\hat{v} - \frac{T_0}{fQ_0R}\tau^{(y)} \\
\hat{u}_x + \hat{v}_y &= Q - \epsilon_2 h
\end{aligned}
\qquad (2.4b)
$$

If we now introduce a stream function Ψ for the baroclinic solution which is geostrophic to lowest order ($\Psi^{(0)} = h$), and arbitrarily assume that $2\epsilon_1 = \epsilon_2 = \epsilon$, the baroclinic vorticity equation is approximately

$$\epsilon \nabla_H^2 h + \beta h_x - \epsilon h = -Q - \frac{}{f Q_o} (\tau_x^{(y)} - \tau_y^{(x)}) \qquad (2.5)$$

The boundary conditions are for no normal flow through any barriers, that is

$$\hat{u} = -h_y - \epsilon \hat{v} = 0 \quad \text{at meridional barriers} \qquad (2.6)$$

$$\hat{v} = h_x + \epsilon \hat{u} = 0 \quad \text{at zonal barriers}$$

(In general, the apparently more rigorous conditions $u_1 = u_2 = 0$ or $v_1 = v_2 = 0$ will be met by solutions with boundary conditions (2.6) as long as no extra conditions are put on the flow.)

III. Meridionally Uniform Flow

i) f plane, $\tau = 0$

We will begin with the simplest possible case: steady, meridionally uniform forcing of the form $Q(x)$ on the f plane with no wind stress, in a channel of width $2L$ centered at $x = 0$. The f plane, y independent vorticity equation which must be satisfied is

$$\epsilon (h_{xx} - h) = -Q \qquad (3.1a)$$

subject to the boundary conditions $u = -\epsilon h_x = 0$ at $x = \pm L$

The simplest subcase of this is uniform sinking everywhere, $Q = A$. The equally simple result is that

$$h = \frac{A}{\epsilon} \qquad (3.2)$$

In other words, the upward motion of the interface due to uniform sinking everywhere is balanced by upward diffusion everywhere. There are no velocities associated with this displacement.

If we allow for x variation in the forcing so that $Q = Ax$, the principle balance in the interior is still between forcing and diffusion (the particular solution to (2.5) is $h = \frac{Ax}{\epsilon}$). However, because the interface is now tilted, meridional geostrophic velocities are generated which have zonal $O(\epsilon)$ velocities associated with them. The resulting solution which satisfies the no normal flow boundary conditions at $x = + L$ is

$$h = \frac{A}{\epsilon} x - \frac{\sinh x}{\cosh L} \tag{3.3}$$

which is the interior particular solution corrected with boundary layers at the two walls. In the neighborhood of the wall $x = L$, the solution (3.3) is of the form $h = \frac{A}{\epsilon}(x - e^{x-L})$ with an exponential boundary layer correction. The dimensional width of the boundary layers is the Rossby radius $R = \frac{\sqrt{g'H'}}{f}$. If we had retained the separate friction and diffusion parameters $2\epsilon_1$ and ϵ_2 from 2.4b), the dimensional boundary layer width would be $\sqrt{\frac{2\epsilon_1}{\epsilon_2}}\frac{\sqrt{g'H'}}{f}$. As the "friction" $2\epsilon_1$ is increased the boundary layer width increases, and as "diffusion" ϵ_2 is increased, the boundary layer width decreases.

The boundary layers result from the deformation of the interface at the wall caused by the nonzero zonal velocities in the interior. As the interface is pushed up or down, geostrophic boundary currents are created which in turn have $O(\epsilon)$ zonal velocities associated with them which oppose the interior zonal flow. A balance is achieved in the boundary layer where the up or downwelling caused by interior zonal velocities is exactly balanced by diffusion. A schematic diagram of this flow is shown in Fig. 2 where the velocities associated with the two parts of the solution are shown separately.

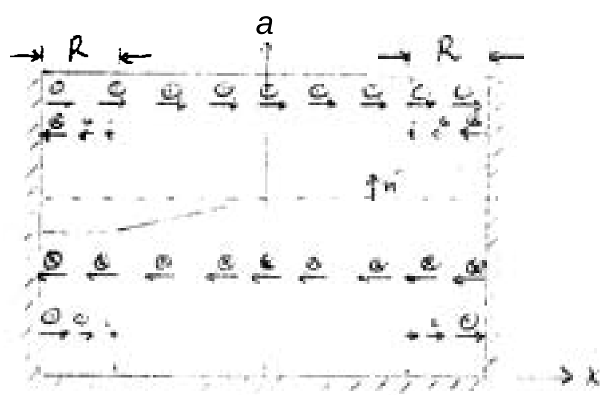

Fig. 2. Cross setion of the flow associated with the sink Q = Ax, f plane. The upper row of velocities in each layer are the velocities associated with the interior solution while the lower row of velocities are associated with the boundary correction.

If we go to more complicated y independent forcing of the flow, the only additional result is that the interior flow gains relative vorticity in addition to an interface displacement. The boundary layer structure remains the same. An example of this is the flow due to the transfer Q = Asin kx which has the full solution

$$h = \frac{A}{\epsilon(1 + k^2)} \left(\sin kx - \cos kL \frac{\sinh x}{\cosh L} \right). \tag{3.4}$$

The interior solution thus has both relative vorticity as well as an interface displacement while the boundary correction is still the familiar exponential correction.

(One further note is that $Q = A\sinh x$ causes a resonant response which no amount of damping can cause to be steady.)

(ii) f plane, $Q = 0$

If a north–south wind blows through the channel an additional component of zonal velocity is induced, namely, the Ekman flux at right angles to the wind. It is not strictly correct to include the wind in this model since there is no bottom friction to damp out the barotropic component of wind induced stress without further damping. Because a nonzero Ekman flux can arise from a uniform wind, we get interior $O(\epsilon)$ velocities for $O(1)$ boundary currents without interior interface displacement. If, for instance, $\underset{\sim}{T} = \hat{j}T_0$, the full solution is

$$
\begin{aligned}
h &= -\frac{T_0 \sinh x}{\epsilon \cosh L} \\
\hat{u} &= -T_0\left(1 - \frac{\cosh x}{\cosh L}\right) \\
\hat{v} &= -T_0 \frac{\cosh x}{\epsilon \cosh L}
\end{aligned}
\qquad (3.5)
$$

in which the zonal velocity $\hat{u} = -T_0$ is compensated by boundary currents in both sides. (The equations have been rescaled with $\epsilon = \frac{T_0}{f^2} \frac{H'}{H}$.)

iii) β plane

Inclusion of variations of the Coriolis parameter allows for the possibility of different vorticity balances, as is well known in studies of steady ocean circulation where interior change in planetary vorticity occurs more readily than interior change in relative vorticity. Inclusion of vortex stretching in steady β plane ocean circulation models can modify boundary layer and possibly interior balances, depending on the magnitude of the diffusion relative to the β effect.

The vorticity equation (2.5) is a steady statement of the potential vorticity equation

$$\frac{D}{Dt}\left(\frac{\nabla_H^2 h + f}{h}\right) = -Q - \epsilon \nabla_H^2 h + \epsilon h \tag{3.6}$$

where we allow for diffusion in addition to frictional dissipation. We can find solutions to (2.5) directly from the equation and boundary conditions or use the Longuet-Higgins transformation to get us back to an f plane type equation which has already been solved for various transfers Q. That is, letting $h(x,y) = \varphi(x,y)e^{\kappa x}$ where $\kappa = -\frac{\beta}{2\epsilon}$ equation (2.5) (without wind stress) becomes

$$\nabla_H^2 \varphi - (\kappa^2 + 1)\varphi = -\frac{Q}{\epsilon} e^{-\kappa x} \tag{3.7}$$

Uniform forcing Q = A yields the same solution as on the f plane, $h = \frac{A}{\epsilon}$.

The balance is still purely between the input of vorticity by the source-sink and diffusion. Linear forcing Q = Ax implies a slightly different particular solution

$$h_p = \frac{A}{\epsilon}\left(x + \frac{\beta}{\epsilon}\right)$$

which is the linear f-plane solution shifted to the west by the β effect. Using (3.7) to obtain a full solution we find that

$$h = \frac{A}{\epsilon}\left(x + \frac{\beta}{\epsilon}\right) + \frac{A}{\epsilon}\left(K\,\frac{\cosh KL}{\cosh\sqrt{K^2+1}\,L} + \sqrt{K^2+1}\,\frac{\sinh KL}{\sinh\sqrt{K^2+1}\,L}\right)e^{Kx}\cosh\sqrt{K^2+1}\,x$$
$$+ \frac{A}{\epsilon}\left(K\,\frac{\sinh KL}{\sinh\sqrt{K^2+1}\,L} + \sqrt{K^2+1}\,\frac{\cosh KL}{\cosh\sqrt{K^2+1}\,L}\right)e^{Kx}\sinh\sqrt{K^2+1}\,x .$$

Clearly, on the β plane it makes more sense to look at approximate solutions in various regions of the basin rather than solving the problem exactly.

The general solution from which (3.8) was obtained is

$$h = c_1 e^{\left(\sqrt{\left(\frac{\beta}{2\epsilon}\right)^2 + 1} - \frac{\beta}{2\epsilon}\right)x} + c_2 e^{-\left(\sqrt{\left(\frac{\beta}{2\epsilon}\right)^2 + 1} + \frac{\beta}{2\epsilon}\right)x} \qquad (3.9)$$

Even without solving explicitly for c_1 and c_2, we can see that near an eastern boundary, the dominant first term will yield a boundary layer width of

$$\frac{1}{\sqrt{\left(\frac{\beta}{2\epsilon}\right)^2 + 1} - \frac{\beta}{2\epsilon}}$$

which is larger than the boundary layer width near a western boundary,

$$\frac{1}{\sqrt{\left(\frac{\beta}{2\epsilon}\right)^2 + 1} + \frac{\beta}{2\epsilon}} .$$

The term $e^{-\frac{\beta}{\epsilon}x}$ skews the entire solution to the west. For $\frac{\epsilon}{\beta} \ll 1$, the x dependence of the eastern and western boundary layer is approximately $e^{\frac{\epsilon}{\beta}x}$ and $e^{-\frac{\beta}{\epsilon}x}$ respectively, illustrating even more simply the shift to the west.

On the f plane, the vorticity balance in both the interior and boundary regions includes relative vorticity, diffusion and forcing, i.e. all of the terms available, except in the case of especially simple forcing. On the β-plane, the extra planetary vorticity term has the effect of allowing different balances in different parts of the basin. If the interface height varies slowly with x, such that $x = \frac{X}{\epsilon}$, the vorticity equation (2.5) becomes

$$\epsilon^3 h_{XX} + \epsilon(\beta h_X - h) = -Q$$

so the dominant balance is clearly

$$\beta h_X - h = -\frac{Q}{\epsilon} \qquad (3.10)$$

whose solution corresponds to a broad eastern boundary layer, dominated by changes in planetary vorticity and vortex stretching.

If $\beta \gg \epsilon$ (2.5) becomes

$$\beta h_x = -Q \qquad (3.11)$$

which is the classical Sverdrup balance. Diffusion is not at all important here and will also not be important in the western boundary layer. The Sverdrup balance can perhaps be thought of as a limiting case of the eastern boundary layer from (3.10).

In regions where the interface height varies rapidly with x, such that $x = \epsilon X$, the vorticity equation (2.5) bcomes

$$h_{XX} + \beta h_X = 0 \qquad (3.12)$$

whose solutions correspond to a narrow western boundary layer, dominated by changes in relative and planetary vorticity.

Thus, with the β term we can match solutions in various regions in addition to trying to solve the problem exactly. For example, the problem solved exactly above with $Q = Ax$ can be solved approximately with (3.10) and (3.12) and found to be

$$h = h_{east} + h_{west} \sim \frac{A}{\epsilon}\left(x + \frac{\beta}{\epsilon}\right) - \frac{A}{\epsilon}\frac{\beta}{\epsilon} e^{\frac{\epsilon}{\beta}(x-L)} - \frac{A}{\beta}\left(1 - e^{-\frac{2\epsilon}{\beta}}\right) e^{-\frac{\beta}{\epsilon}(x+L)}$$

a substantial simplification of (3.8) for $\frac{\epsilon}{\beta}$ 1.

On both the f and β planes we see that eastern and western boundary layers occur whenever the interior zonal velocity is nonzero, whether it is forced directly as an Ekman flux by the wind or more indirectly as a result of the geostropic flow due to divergences created by mass transfer or wind stress curl. Steady solutions with this boundary layer structure are possible only because of the diffusion term ϵh which allows a damped form of vortex stretching to occur in steady flow.

With the structure of solutions on the f and β planes for meridionally uniform flow in mind, we move to forcing which may vary with latitude.

IV. Zonally Uniform Forcing in a Meridional Channel

What is the result of cooling which varies with latitude? As a simple case, we will consider cooling which is uniform and positive (transfer to the lower layer) in a northern basin and zero to the south with a transition region between which is wider than the Rossby radius but not as wide as $\frac{1}{\epsilon}$. (We could equally well choose any uniform value for the two halves of the basin.) We have in mind an enclosed sea like the Norwegian Sea but make the simplifications that a) the basin length is greater than $\frac{1}{\epsilon}$ and b) curvature of the basin occurs on a scale larger than the Rossby radius.

The geometry we are considering is illustrated in Fig. 3. For simplicity we will assume that $Q = A$ for $y > 0$, $Q = 0$ for $y \le -M$ and $Q = A(1 + \frac{y}{M})$ in the transition region. This will necessarily lead to discontinuity in the zonal velocity at $y = 0, -M$ but is simple to solve, and has the essential features of a solution with smoother forcing.

The vorticity equation becomes

$$\epsilon(h_{xx} + h_{yy}) - h = -Q \tag{4.1a}$$

subject to the boundary conditions

$$u = -h_y - \epsilon h_x = 0 \text{ at } x = \pm L$$

$$h = 0 \qquad \text{at} \quad y = -M \tag{4.1b}$$

$$h \text{ continuous at } y = 0$$

We consider the solution in three stages: i) solution for $-M \leq y < 0$ with no x dependence, ii) solution near $x = \pm L$ with large scale variation in y and iii) matching the solutions in the corner near $x = L$, $y = 0$. It will be seen that variation in forcing Q with latitude allows Kelvin wave-like disturbances to be found far to the north in the region of uniform Q which would otherwise be undisturbed. Thus we may in some way be able to model a more global forcing of the eastern boundary current in the Norwegian Sea than we could otherwise obtain with local winds and forcing.

i) The solution for $-M \leq y \leq 0$ with $\frac{\partial}{\partial x} = 0$ is just the particular solution

$$h = \frac{A}{\epsilon}\left(1 + \frac{y}{M}\right) \tag{4.2}$$

which has a zonal geostrophic velocity $u = -\frac{A}{\epsilon M}$ and meridional Ekman velocity $\hat{v} = -\epsilon h_y = -\frac{A}{M}$ associated with it. Thus for $A > 0$, there is an eastward (and northward) density current in the upper layer and the opposite in the lower layer.

ii) In the regions where $Q = A$ and $Q = 0$, we stretch the y coordinate by ϵ so that $y = \frac{Y}{\epsilon}$. The vorticity equation (4.1a) is then approximately (to $O(\epsilon)$)

$$h_{xx} - h = -\frac{Q}{\epsilon}$$

subject to the boundary conditions

$$\hat{u} = -\epsilon(h_Y + h_x) = 0 \text{ at } x = L.$$

Letting

$$h = \frac{A}{\epsilon} + C_1(Y)e^{x-L} + C_2(Y)e^{-(x+L)}$$

the only solution which decays away to the north, where Q = A, is

$$h = \frac{A}{\epsilon} + C_1 e^{-Y+x-L}$$

and likewise, the solution which decays away to the south, where Q = 0, is

$$h = c_2 e^{Y-(x+L)}$$

if there were a wall to the south at x = -L. These correspond to boundary layers with width equal to the Rossby radius Ro and length of R_o , decaying away from the region of varying Q. Considering only the solution to the north, the constant c_1 can be determined by matching transports in the boundary layer at y = 0 with the eastward transport in $-M < y < 0$. We thus obtain

$$h = \frac{A}{\epsilon}\left(1 - e^{-\epsilon y + x - L}\right) \qquad y > 0 \qquad (4.3)$$

for the boundary layer decaying away to the north.

iii) In the corner near y = 0, x = L, the vorticity equation is

$$h_{xx} + h_{yy} - h = -\frac{A}{\epsilon}\left(1 + \frac{y}{M}\right)$$

subject to the boundary condition

$$\hat{u} = -h_Y = 0 \qquad \text{at } x = L$$

$$h = 0 \qquad \text{at } y = -M$$

$$h = \frac{A}{\epsilon}\left(1 - e^{x-L}\right) \qquad \text{at } y = 0$$

The vorticity equation is separable and the solution which fits the boundary conditions is

$$h = \frac{A}{\epsilon}\left(1 + \frac{y}{M}\right)\left(1 - e^{x-L}\right) \qquad -M < y < 0 \qquad (4.4)$$

The full solution for the basin is

$$h = \begin{cases} \frac{A}{\epsilon}\left(1 - e^{-\epsilon y + x - L}\right) & y > 0 \\ \frac{A}{\epsilon}\left(1 + \frac{y}{M}\right)\left(1 - e^{x-L}\right) & -M < y < 0 \\ 0 & y < -M \end{cases} \qquad (4.5)$$

We note particularly that the zonal velocity is identically zero for $y > 0$ (a characteristic of a Kelvin wave) and that there is a nonzero geostrophic meridional velocity along the eastern coast for $y > 0$ which is solely due to the variation in Q from $-M < y < 0$. The upper layer velocities associated with the interface displacement (4.5) are shown schematically in Fig. 3.

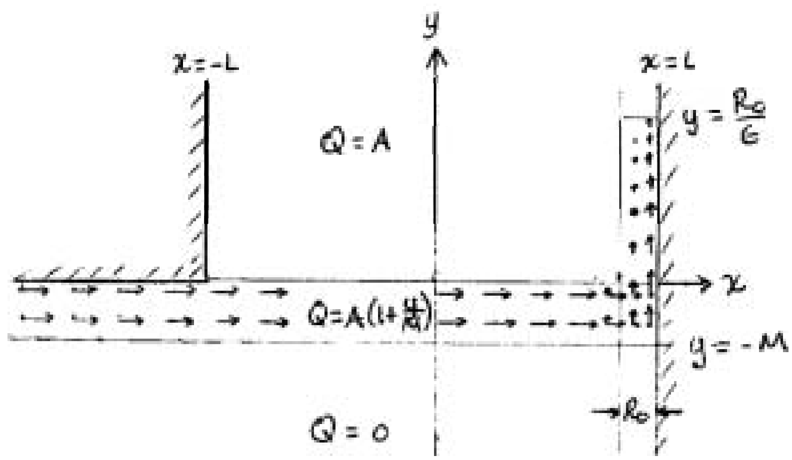

Fig. 3. Geometry and velocities when $Q = Q(y)$: specifically, $Q = A$ for $y > 0$, $Q = A(1 + \frac{y}{M})$ for $-M < y < 0$ and $Q = 0$ for $y < -M$.

It appears then that the steady signature of the Kelvin wave, which would arise in the time dependent case and travel up the eastern coast to the north, is a boundary current which eventually damps out to the north due to friction. Therefore, even in a region where variation in forcing is too weak to provoke a flow, there can be flow due to variation in the forcing elsewhere (if variation is on a scale less than $\frac{1}{\epsilon}$). The mechanism is quite simple: the eastward flowing density current (upper layer) reaches the wall and causes a downward displacement of the interface which eventually reaches a diffusive equilibrium. Geostrophic velocities to the north (upper layer) along the boundary result and damp out on the frictional scale $\frac{1}{\epsilon}$. (The Ekman velocity associated with the geostrophic velocity in the region $-M < y < 0$ balances the incoming flow so that it is zero at the wall). Zonal Ekman velocities in the northward extension which are generated by the geostrophic northward velocities are exactly balanced by geostrophic zonal velocities due to the variation in interface height with latitude y (as a result of damping).

Extension of these results to the β plane is quite simple and involves expanding the eastern boundary layer width by $\frac{\epsilon}{\beta}$ (for $\frac{\epsilon}{\beta} \ll 1$), the familiar skewing of the circulation to the west. Damping to the north also occurs over a scale expanded by $\frac{\epsilon}{\beta}$. If we think of the boundary current as a damped Kelvin wave, the extension of the layer to the west can perhaps be thought of as damped nondispersive Rossby waves.

Combination of variation of forcing with both latitude and longitude can thus give rise to wide eastern boundary currents and narrow western boundary currents. Forcing in the Norwegian-Greenland Sea is irregular but the general trend is for much higher heat flux to the north (Bunker and Worthington, 1976) and a general cyclonic wind stress pattern. The wide northward flowing eastern boundary current which is observed in the southern Norwegian Sea (Metcalf, 1960) may be the result of latitudinal variation in cooling and/or wind stress south of the entrance to the Norwegian Sea, local northward or cyclonic wind stress, or local cooling. We note that an eastern boundary current at a particular latitude can be caused only by 1) local forcing which produces a zonal flow that must be compensated at the boundaries or 2) variations in forcing to the south of that latitude which produces a damped Kelvin wave northward of the variation. Therefore, variations in facing or forcing itself to the north of that latitude have no influence on the eastern boundary current there.

V. Point Transfer on the f and β planes

Bottom Water formation may occur Locally and sporadically near the center of cyclonic gyres where the stratification is weakest due to doming of the previously formed Bottom Water. It may be possible to model some aspects of the flow due to Bottom Water formation with the steady model considered so far. For this purpose, we will simply assume that mass and momentum transfer is a delta function and look for steady solution. No account is taken of wind stress, preconditioning, or the spin up or spin down which must undoubtedly occur with a time dependent process. Modification of the flow by the β effect is considered. In reality, the Greenland gyre may be very strongly influenced by topography since it appears to sit squarely in the Greenland Basin.

On the f plane, the vorticity equation for a point source $Q = \frac{\delta(r)}{r}$ with no angular dependence is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) - h = -\frac{\delta(r)}{\epsilon r} \tag{5.1}$$

which has solutions $K_0(r)$ and $I_0(r)$ with the jump condition

$$\left[r\frac{\partial h}{\partial r}\right] = -\frac{1}{\epsilon} \qquad \text{at} \quad r = 0 .$$

Choosing the exponentially decaying solution $h = AK_o(r)$, the interface height and azimuthal and radial velocities for large r are

$$\begin{aligned} h &\sim A\sqrt{\frac{\pi}{2r}}\, e^{-r} \\ \hat{u}_\theta &\sim \frac{\partial h}{\partial r} \sim -A\sqrt{\frac{\pi}{2r}}\, e^{-r} \\ \hat{u}_r &\sim -\epsilon \hat{u}_\theta \sim \epsilon A \sqrt{\frac{\pi}{2r}}\, e^{-r}\, . \end{aligned} \tag{5.2}$$

This corresponds to outward velocity, clockwise rotation and cyclonic vorticity in the lower layer and the opposite in the upper layer, as illustrated in Fig. 4. Cyclonic vorticity in the lower layer arises from point vortex stretching at r = 0. The vorticity decreases away from the center as the water parcels move outward and are squashed. Clockwise rotation in the lower layer clearly arises from outward motion of water parcels in a counterclockwise rotating system due to angular momentum conservation.

On the β plane, the flow is skewed to the west as expected. Using the Longuet Higgins transformation, the interface height is easily seen to be

$$h = AK_o\left(\sqrt{\kappa^2 + 1}\; r\right) e^{\kappa x} \quad \text{where} \quad \kappa = -\frac{\beta}{2\epsilon}\, .$$

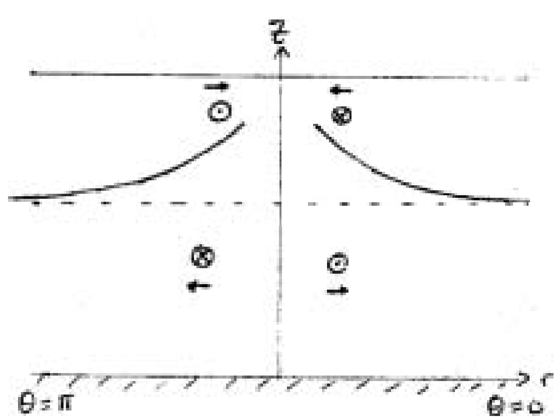

Fig. 4. Interface height h and velocities associated with $Q = \frac{\delta(r)}{r}$ on the f plane.

For large r, this becomes

$$h \sim A \sqrt{\frac{\pi}{2\sqrt{K^2+1}\, r}}\; e^{r(-\sqrt{K^2+1} + K\cos\theta)}$$

If $|K| = \frac{\beta}{2\epsilon}$ is sufficiently large, we have

$$h \sim A \sqrt{\frac{\pi\epsilon}{\beta r}}\; e^{r\frac{\beta}{2\epsilon}(1-\cos\theta)}$$

The loci of constant phase $r(1-\cos\theta)$ are parabolas which open to the west (Rhines, 1979 lecture notes) and mass transfer is clearly predominantly to the west.

Thus, on both the f and β planes, a point transfer of mass and momentum generates a steady counterclockwise flow in the upper layer and clockwise flow in the lower layer with the highest velocities near the transfer point. This type of flow in the upper layer accords with observations of counterclockwise flow in the Greenland Sea (Metcalf, 1960). It is, however, not to be forgotten that the wind stress in this region also yields a counterclockwise gyre, so perhaps the effects reinforce each other in the production of the gyre and in Bottom Water formation.

VI. Summary

Inclusion of a representation of diffusion in the continuity equation appears to be a useful way of damping the circulation resulting from a steady transfer of mass and momentum from one layer to another. With this term included in the vorticity equation it is possible to meet boundary conditions on the flow with boundary layers since the interface displacement at the boundaries can be an equilibrium between upwelling and diffusion.

The usual steady circulation models on the β plane cannot have eastern boundary layers because there is no way to balance relative vorticity accession and changes in planetary vorticity on the eastern boundary. It is for this reason that boundary layers occur only in the west while the balance elsewhere is between forcing and changes in planetary vorticity (Sverdrup balance), in the usual β plane models. Inclusion of damping of vortex stretching in the form of the diffusion term ϵh in the vorticity equation allows the presence of western and eastern boundary layers. The western boundary layer still has the same structure as before, but the interior (eastern) solution includes diffusion as well, as long as $\frac{\epsilon}{\beta}$ is not too small.

Application of these results to the actual flow in the Norwegian-Greenland Sea may be somewhat tenuous but two features deserve mention. The first is the broad northward Norwegian Current which may possibly be modelled as the northern damped Kelvin wave extension of an eastward density or wind driven current in the northern North Atlantic. The second is the counterclockwise circulation in the Greenland Basin which may be partially driven by the formation of Bottom Water at its center and may be roughly modelled by the point transfer of Topography and wind may play a very large role in determining the actual circulation, which itself probably undergoes large seasonal variation.

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REFERENCES

Bunker, A. F. and L. V. Worthington, 1976. Energy exchange charts of the North Atlantic Ocean. Bull. Am. Meteor. Soc., 57, 670–678.

Carmack, E. and K. Aagard, 1973. On the deep water of the Greenland Sea. DSR, 20, 687–715.

Gill, A. F., J. M. Smith, R. P. Cleaver, R. Hide and P. R. Jonas, 1979. The vortex created by mass transfer between layers of a rotating fluid. Geophy. and Astrophy. Fluid Dynamics, 12, 195–220.

Gill, A., 1979. GFD Lectures.

Rhines, P., 1979. Class notes from "Mesoscale Ocean Dynamics".